Entropy viscosity method for high-order approximations of conservation laws

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Abstract A stabilization technique for conservation laws is presented. It consists of introducing in the governing equations a nonlinear dissipation function of the residual of the associated entropy equation and bounded from above by a first order viscous term. Different two-dimensional test cases are simulated - a 2D Burgers problem, the "KPP rotating wave" and the Euler system - using high order methods: spectral elements or Fourier expansions. Details on the tuning of the parameters controlling the entropy viscosity are given.

1 Introduction

High-order methods, especially spectral methods, are very efficient for solving Partial Differential Equations (PDEs) with smooth solutions since the approximation error goes exponentially fast to zero as the polynomial degree of the approximation goes to infinity, *i.e.* spectral accuracy is observed. Unfortunately this property breaks down for non-smooth solutions such as those that arise from solving nonlinear conservation laws. This type of equations generates shocks which in turn induce the so-called Gibbs phenomenon. The problem is not new and many sophisticated algorithms have been developed to address this issue. Particularly popular among these methods are the so-called monotone and Total Variation Diminishing (TVD) schemes that aim at enhancing the accuracy far from the shocks and promoting non-

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oscillatory behavior at the shocks. These techniques are mainly based on Essentially Non Oscillatory polynomial reconstructions (ENO) and the use of flux/slope limiters whose goal is to bound the fluxes. One may consult [3] for an overview on this class of methods, which were mainly developed for Finite Volume approximations.

It is remarkable that few methods have been proposed for solving nonlinear conservation laws with high order methods. Among them, in the frame of spectral methods the well known "spectral vanishing viscosity" [6] technique consists of introducing a dissipation term only active in the high frequency range of the spectral approximation. In the same spirit, but on the basis of a hp-finite element approximation and a Discontinuous Galerkin method, it was also recently proposed to introduce a dissipation term, based on a viscosity controlled by a smoothness indicator [5]. The goal of the present paper is to present a somewhat different viscosity method, which was recently introduced in [2] by the authors. Here again the key idea consists of augmenting the PDE with a dissipation term, but the viscosity is based on the residual of the associated entropy equation. Here we propose a simplified formulation of the method and extend it to two-dimensional problems. The technique is implemented with Fourier polynomials and the Spectral Element Method (SEM).

The paper is organized as follows. We describe the entropy viscosity method in Section 2. An application to the two-dimensional inviscid Burgers equation with Fourier polynomials is described in Section 3 and convergence tests are reported. The method is adapted to the SEM setting in Section 4 and is illustrated on a nonlinear conservation law exhibiting a rotating composite wave. In Section 5 we adapt the entropy viscosity method to the two-dimensional Euler system and solve a classical benchmark problem using the Fourier approximation.

2 The entropy viscosity method

It is well known that the relevant weak solution of the *scalar* conservation law

$$\partial_t u(\mathbf{x},t) + \nabla \cdot \mathbf{f}(u(\mathbf{x},t)) = 0\,, \quad \mathbf{x} \in \Omega\,, \quad t \in \mathbb{R}^+ \tag{1}$$

with appropriate initial and boundary conditions, is the so-called entropy solution, which is also characterized by $u = \lim_{\nu \to 0} u_\nu$ where

$$\partial_t u_\nu + \nabla \cdot \mathbf{f}(u_\nu) = \nu \Delta u_\nu. \tag{2}$$

Let us recall the following points, see *e.g.* [3] and references herein: a) Solving (2) rather than (1), with a "small" value of ν, yields the Von-Neumann-Richtmyer method, developed for the Euler equations in 1950 ! Such an approach is however well known to be too diffusive. b) Linear techniques such as the Lax-Wendroff scheme are more accurate than the first-order viscosity regularization but they are not fully satisfactory since the solution is often polluted by spurious oscillations. To overcome this difficulty one usually resorts to TVD schemes. c) High-order (> 1) TVD (and so monotonicity preserving) schemes must be nonlinear, as stated by the

Godunov theorem. d) Nonlinear schemes with flux/slope limiters essentially consist of adding some *nonlinear viscosity dissipation*.

Starting from this last point, the entropy viscosity method is based on introducing a nonlinear dissipation term $\nabla.(\nu_h \nabla u)$ in the right hand side of (1). Let $E(u)$ be a convex function and assume that there exists an entropy pair $(E(u),\mathbf{F}(u))$ such that

$$\partial_t E(u) + \nabla \cdot \mathbf{F}(u) \leq 0$$

characterizes the unique viscous limit to (1) (*i.e.* the entropy solution). Let $r_E(u) := \partial_t E(u) + \nabla \cdot \mathbf{F}(u)$ be the entropy residual. This quantity is a negative measure supported on the shocks, *i.e.* $r_E < 0$ at the shocks and $r_E = 0$ elsewhere.

Assume that the computational domain Ω is discretized, let h be the grid size and u_h the numerical solution. We propose to construct a local artificial nonlinear viscosity based on the entropy residual $r_E(u_h)$. To this end we first set

$$\nu_E(\mathbf{x},t) := \alpha h^2(\mathbf{x}) \mathscr{R}(r_E(u_h))/\|E(u_h) - \bar{E}\|_{\infty,\Omega} \tag{3}$$

where α is a proportionality coefficient, $\bar{E}$ is the space average of $E(u_h)$ (recall that E is defined up to a constant), $\|.\|_{\infty,\Omega}$ is the usual $L^\infty(\Omega)$ norm and $\mathscr{R}(r_E)$ is a positive function (or functional) of the residual r_E. The terms $h^2(\mathbf{x})$ and $\|E(u_h) - \bar{E}\|_{\infty,\Omega}$ are scaling factors. The aim of $\mathscr{R}(r_E)$ is to extract a useful information from the residual; Hereafter we use $\mathscr{R}(r_E) = |r_E|$. Note that in smooth parts of u, one may expect that $r_E(u_h)$ scales like the approximation error of the resolution method.

We now provide an upper bound for the entropy viscosity. For the one-dimensional scalar conservation equation $\partial_t u + f'(u)\partial_x u = 0$, the first-order Finite Difference upwind scheme (linear monotone scheme) is equivalent to the second-order centered Finite Difference augmented with a viscous dissipation with viscosity $\nu_{max} = \frac{1}{2} f'(u) h$. By analogy we set

$$\nu_{max}(\mathbf{x},t) = \alpha_{max}\, h \max_{\mathbf{y}\in V_\mathbf{x}} |\mathbf{f}'(u_h(\mathbf{y},t))|, \tag{4}$$

where α_{max} is a constant coefficient, and $V_\mathbf{x}$ is a neighborhood of $\mathbf{x}$ still to be defined and dependent on the approximation method. In practice the size of $V_\mathbf{x}$ is a few multiples of h in each direction. Finally the entropy viscosity is defined to be

$$\nu_h(\mathbf{x},t) := \mathscr{S}(\min(\nu_{max}, \nu_E)) \tag{5}$$

where $\mathscr{S}$ is a smoothing operator. Smoothing may indeed be required because $r_E(u_h)$ is generally highly oscillatory, since when a shock occurs we actually try to approximate a Dirac distribution. Practical implementation details on the operator $\mathscr{S}$ and on the neighborhood $V_\mathbf{x}$, as well as details on how to tune the coefficients α and α_{max} are provided in the examples studied in next sections.

3 2D Burgers (Fourier)

Let $\Omega = (0,1)^2$ and consider the following inviscid Burgers problem, where $\mathbf{v} = (1,1)$ is a constant vector field:

$$\partial_t u + \nabla.(\frac{1}{2}u^2\mathbf{v}) = 0, \qquad u|_{t=0} = u_0(x,y) \tag{6}$$

where $u_0 = -0.2$ if $x < 0.5$, $y > 0.5$, $u_0 = -1$ if $x > 0.5$, $y > 0.5$, $u_0 = 0.5$ if $x < 0.5$, $y < 0.5$ and $u_0 = 0.8$ if $x > 0.5$, $y < 0.5$. The local velocity $\mathbf{f}'(u) = u\mathbf{v}$ is parallel to $\mathbf{v}$ and of amplitude u.

To be able to solve this problem with Fourier expansions we transform it into a periodic problem by extending the computational domain to $(0,2)^2$ and by extending the initial condition by symmetry about the axes $\{x = 1\}$, $\{y = 1\}$.

We choose the entropy pair $E(u) = \frac{1}{2}u^2$, $\mathbf{F}(u) = \frac{1}{3}u^3\mathbf{v}$, and then follow the procedure described in Section 2. The entropy viscosity is computed in the physical space at the Fourier nodes. For each Fourier node $\mathbf{x}$, the neighborhood $V_\mathbf{x}$ is composed of the 7×7 Fourier nodes surrounding $\mathbf{x}$. The smoothing operation is performed by doing two smoothing sweeps, each one based on a two-dimensional averaging rule involving 5 grid-point values, with weight 4 for the central point and 1 for the 4 closest points.

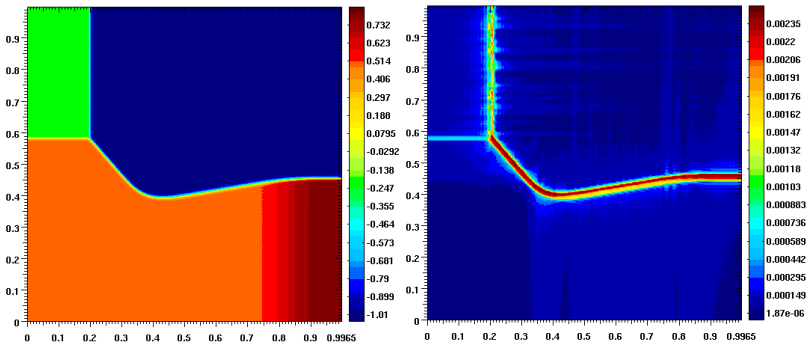

Fig. 1 Fourier-RK4 solution (Left) and entropy viscosity (Right).

The time marching is done by using the standard Runge-Kutta scheme (RK4). The entropy viscosity is taken constant in time during the time-step, say from time t_n to t_{n+1}, and so computed at time t_n. Using the second order backward finite difference approximation, the time derivative of the entropy is computed from the values of E at time t_n, t_{n-1} and t_{n-2}.

We show in Fig. 1 computations done at time $t = 0.5$ with 192 Fourier modes in each direction, *i.e.* with 192^2 grid points in $(0,1)^2$. The non-linearity was de-aliased using the $\frac{3}{2}$ padding rule. The entropy viscosity control parameters are $\alpha = 0.2$ and

$\alpha_{max} = 1.5$. The approximate solution is shown in the left panel of Fig. 1, and the entropy viscosity is shown in the right panel. The shocks are well described and the entropy viscosity focuses in the shocks as expected.

Table 1 Errors and convergence rates for the 2D Burgers problem.

h	L^1	rate	L^2	rate	L^∞
2.78E-2	1.92E-2	–	1.02E-1	–	1.47
1.39E-2	9.99E-3	0.94	7.28E-2	0.49	1.50
6.94E-3	5.34E-3	0.89	5.41E-2	0.43	1.50
3.47E-3	2.79E-3	0.95	3.80E-2	0.51	1.51

The exact solution to (6) can be evaluated at time $t = 0.5$. Table 1 gives the relative error in the L^1- and L^2-norm for different grid sizes. One observes convergence rates close to optimality, *i.e.* order one in the L^1-norm and half order in the L^2-norm. Of course, no convergence is obtained in the L^∞ norm.

4 KPP rotating wave (SEM)

We now use the SEM method to solve the following two-dimensional nonlinear scalar conservation law:

$$\partial_t u + \nabla \cdot \mathbf{f}(u) = 0, \quad \mathbf{f}(u) = (\sin u, \cos u), \quad u|_{t=0} = \begin{cases} 3.5\pi & \text{if } |\mathbf{x}| < 1 \\ \frac{1}{4}\pi & \text{otherwise} \end{cases}$$

in the domain $\Omega = (-2,2)\times(-2.5,1.5)$ for $t \in (0,1)$. This problem has been proposed by Kurganov, Petrova and Popov [1] to test the convergence properties of some WENO schemes.

The local velocity is $\mathbf{v} = \mathbf{f}'(u) = (\cos u, -\sin u)$. We choose the entropy pair $E(u) = \frac{1}{2}u^2$, $\mathbf{F}(u) = (u\sin u + \cos u, u\cos u - \sin u)$. Then we follow the procedure defined in Section 2.

The domain is uniformly discretized using squares of side h and the approximation space is composed of the functions that are continuous and piecewise polynomial of partial degree at most N. The local shape functions are the Lagrange polynomials associated with the $(N+1)^2$ Gauss-Lobatto-Legendre (GLL) points. To define the entropy viscosity we follow the procedure described in Section 2, except that in (4) we have used the local grid size of the GLL mesh, say h_{GLL}, rather than h. The neighborhood $V_{\mathbf{x}}$ is defined as the corresponding spectral element of $\mathbf{x}$, during the assembling procedure. The smoothing is achieved inside each element on the GLL mesh, by one smoothing sweep based on a two-dimensional averaging rule involving 5 GLL grid-points. The entropy viscosity control parameters are $\alpha = 40$ and $\alpha_{max} = 0.8/N$. The time marching is done by using the standard Runge-Kutta

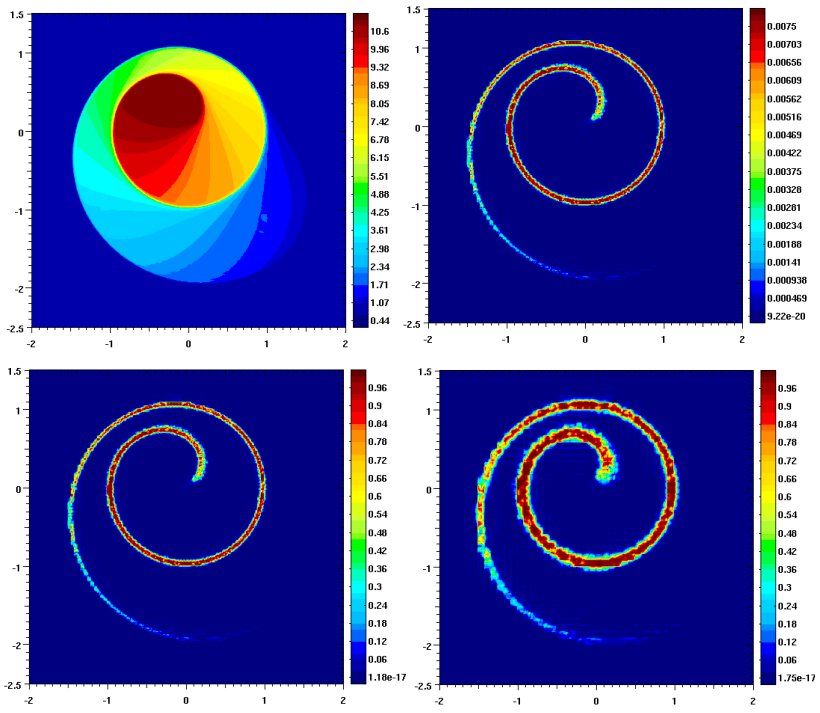

Fig. 2 SEM-RK4 solution for the KPP rotating wave. Solution (Top left) and corresponding entropy viscosity (Top right) for $N = 4$ and 96^2 cells. Ratio ν/ν_{max} for 96^2 cells (Bottom left) and 48^2 (Bottom right).

scheme (RK4). The entropy viscosity is made explicit and computed by using the second order backward finite difference approximation for the time derivative of the entropy.

Results reported in the two top panels of Fig. 2 have be obtained with a grid composed of 96^2 square elements and with polynomials of degree $N = 4$ in each variable. The numerical solution is shown in the left panel; It exhibits the correct composite wave structure. The corresponding entropy viscosity is shown in the right panel; As expected, dissipation is added only where the shock develops.

We finish this section by providing more details on how to adjust the entropy viscosity parameters. The idea is that to be efficient, the viscosity must reach its maximum value in the shocks. Consequently, we propose the following two-step adjustment procedure:

1. Set $\alpha = \infty$ and increase α_{max} until obtaining a smooth solution (a good guideline is that $\alpha_{max} = \frac{1}{2}$ is the correct answer in one space dimension on uniform grids).
2. Once α_{max} is fixed, set α so that the entropy viscosity saturates in the shocks, *i.e.* $\max(\nu) = \nu_{max}$ in shocks.

The two bottom panels in Fig. 2 show the ratio ν/ν_{max} for two different discretizations. Observe that this ratio equals 1 in the shock.

5 2D Euler system (Fourier)

We finish this paper by explaining how the entropy viscosity method can be adapted to the compressible Euler equations:

$$\partial_t \mathbf{u} + \nabla \cdot \mathbf{f}(\mathbf{u}) = 0, \quad \mathbf{u} = \begin{pmatrix} \rho \\ \rho \mathbf{v} \\ E \end{pmatrix}, \qquad \mathbf{f} = \begin{pmatrix} \rho \mathbf{v} \\ \rho \mathbf{v} \otimes \mathbf{v} + p\mathbb{I} \\ \mathbf{v}(E+p) \end{pmatrix} \tag{7}$$

where $p = \rho T$, $T = (\gamma - 1)(E/\rho - \mathbf{v}^2/2)$. Usual notations are used: $\rho, \mathbf{v}, p, T, \gamma, E$ stand for density, velocity, pressure, temperature, ratio of specific heat, and total energy, respectively. The physical entropy functional $S(p,\rho) = \frac{\rho}{\gamma-1}\log(p/\rho^\gamma)$ is such that $r_S := \partial_t S + \nabla \cdot (\mathbf{v}S) \geq 0$.

To understand where and how the entropy dissipation must be set, it is helpful to follow the physics by considering the viscous fluxes appearing in the Navier-Stokes equations:

$$\mathbf{f}_{visc}(\mathbf{u}) = \begin{pmatrix} 0 \\ -\mu \nabla \mathbf{v} \\ -\mu \mathbf{v} : \nabla \mathbf{v} - \kappa \nabla T \end{pmatrix}.$$

The quantity μ is the dynamic viscosity and κ is the thermal conductivity.

First, we compute μ_S, except that there is no need to normalize by $\|S - \bar{S}\|_{\infty,\Omega}$ in (5): $\mu_S = \alpha h^2 \rho(\mathbf{x},t)|r_S(\mathbf{x},t)|$. Then, estimating the maximum local wave speed to be $|\mathbf{v}| + \sqrt{\gamma T}$, we set $\mu_{max} = \alpha_{max} h \rho(\mathbf{x},t) \max_{\mathbf{y} \in V_\mathbf{x}}(|\mathbf{v}(\mathbf{y},t)| + \sqrt{\gamma T(\mathbf{y},t)})$. Finally, $\mu = \mathscr{S}(\min(\mu_{max}, \mu_S))$ and, taking κ to be proportional to μ, $\kappa = \beta\mu$.

We now validate this approach by solving the benchmark problem number 12 from [4]. It is a two-dimensional Riemann problem set in $\mathbb{R}^2$. In the restricted computational domain $(0,1)^2$ the initial set of data is defined as follows:

$$\begin{array}{llllll} p = 1., & \rho = 0.8, & \mathbf{v} = (0.,0.), & 0. < x < 0.5 & 0. < y < 0.5, \\ p = 1., & \rho = 1., & \mathbf{v} = (0.7276,0.), & 0. < x < 0.5, & 0.5 < y < 1., \\ p = 1., & \rho = 1., & \mathbf{v} = (0.,0.7276), & 0.5 < x < 1., & 0. < y < 0.5, \\ p = 0.4, & \rho = 0.5313, & \mathbf{v} = (0.,0.) & 0.5 < x < 1., & 0.5 < y < 1.. \end{array}$$

The solution is computed at time $t = 0.2$. Proceeding as in Section 3, the problem is first made periodic by extending the computational domain to $(0,2)^2$, and the initial data are extended by symmetry about the axes $\{x = 1\}$ and $\{y = 1\}$. The time marching algorithm, the definition of the smoothing operator, and the neighborhood $V_\mathbf{x}$ are the same as in Section 3. The nonlinear terms are de-aliased. The control parameters for the entropy viscosity are $\alpha = 20$, $\alpha_{max} = 0.5$ and $\beta = 2$. We

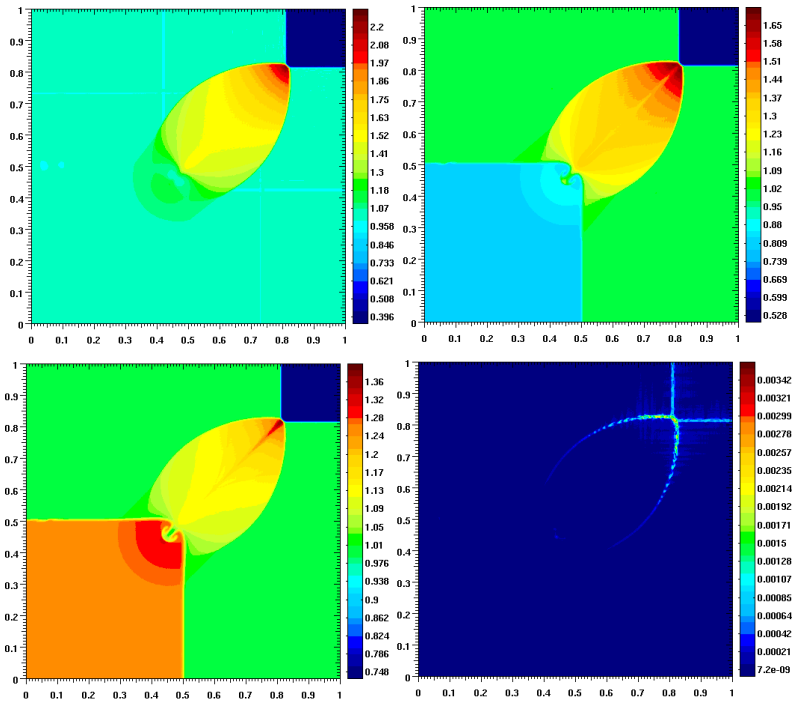

Fig. 3 Pressure (top left); Density (top right); Temperature (bottom left); Entropy viscosity μ (bottom right).

show in Fig. 3 results obtained with 400 Fourier modes in each direction, *i.e.* with 400 grid-points in $(0,1)^2$. They compare well with those obtained with other more sophisticated shock capturing methods, see [4].

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